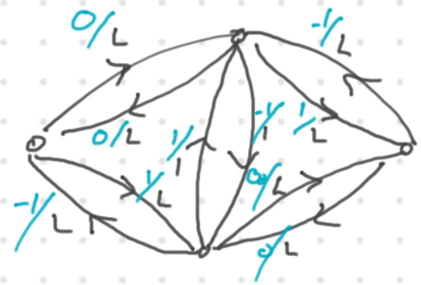
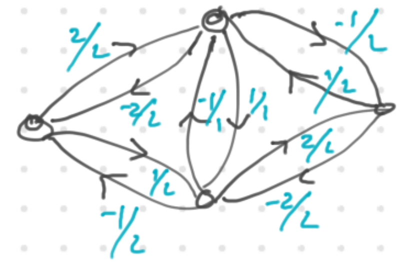
Ford - Folkerson max-flow algorithm



after augmenting $\bar{o}$ on path P?

now we augment the flow by 2.



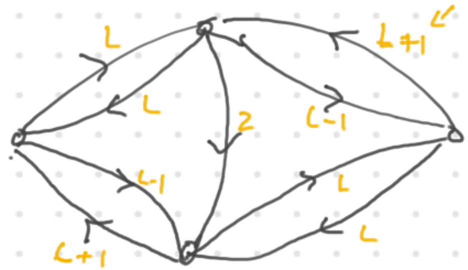
pick directed path P from $s \to t$ in residual graph + find $\alpha = \min_{(u,v) \in E(P)} c(u,v) - f(u,v)$

value of the flow is 3

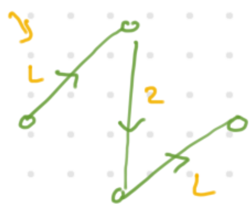
in ex, $\alpha = 1$

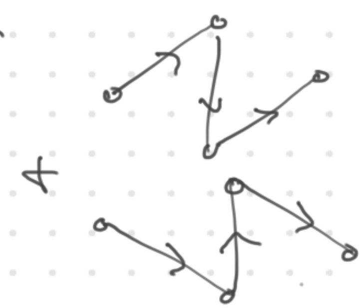
The residual graph is now

we can keep going back + forth augmenting on the paths



$c(u,v) - f(u,v)$





augmenting the flow by 2 at each step.

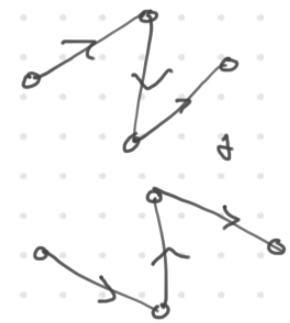
So the algorithm takes $O(L \cdot (n+m))$ steps to terminate (because we're doing $L/2$ augmentations to the flow.)

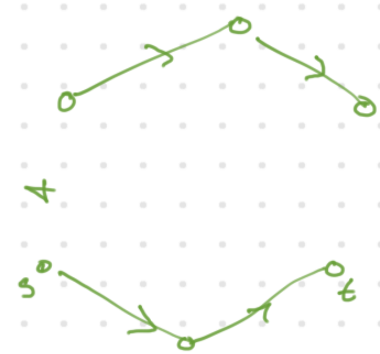
In general, ~~if~~ in a max-flow min cut problem with integer capacities, the FF algorithm has complexity $O(L \cdot (n+m))$ where L is the value of the flow.

Not Polynomial in size of input. Because L ~~is~~ is a function of the capacities + in input it is $\log(L)$ bits to write the capacities. $\Rightarrow$ input has size

$\log(L)(n+m)$

In example from the previous slide, we take $O(L)$ iterations to find the max flow because of the specific paths we chose





augment by L on this path

augment by L on this path

+ doing so arrives a max flow in just two ~~steps~~ augmentations.

Alg (Edmonds Karp) same alg as F.F., but when selecting path in the residual graph to augment on, we pick a shortest s-t path in residual graph.

Note that this is what our optimal choice of paths does in the example - pick paths of length two from $s \to t$ in the residual graph.

Lemma s, t vertices in G w/ capacities on the edges + let $G_1, \ldots, G_k$ be the series of residual graphs in k iterations of Edmonds-Karp algorithm.

$\forall u \in V(G)$ $\forall i$

$\text{dist}_{G_i}(s, u) \leq \text{dist}_{G_{i'}}(s, u)$

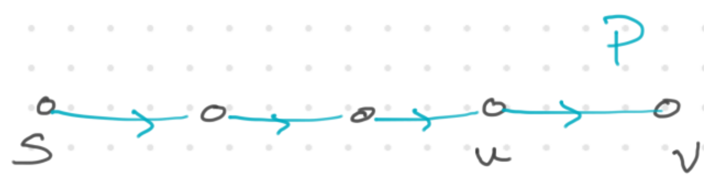
$\forall i' \geq i$

i.e distance from $s \to u$ in residual graph is monotonically increasing.

pf assume false + pick a vertex v + index i st $\text{dist}_{G_i}(s, v) < \text{dist}_{G_{i+1}}(s, v)$

+ from all such vertices v, we pick v to minimize $dist_{G_i}(s, v)$

Look at G_{i+1} + let P be a min length path from $s \to v$ in G_{i+1}.



(in G_{i+1})

let u = vertex immediately before v on P.

$$dist_{G_{i+1}}(s, u) \geq dist_{G_i}(s, u) \quad \text{el} \quad (u, v) \notin E(G_i)$$

by choice of v to be as close as possible to s + not satisfy the statement of the lemma ⇒ lemma holds for u.

if it were

$$dist_{G_{i+1}}(s, u) \geq dist_{G_i}(s, u)$$

$$\| $$

$$dist_{G_{i+1}}(s, v) - 1$$

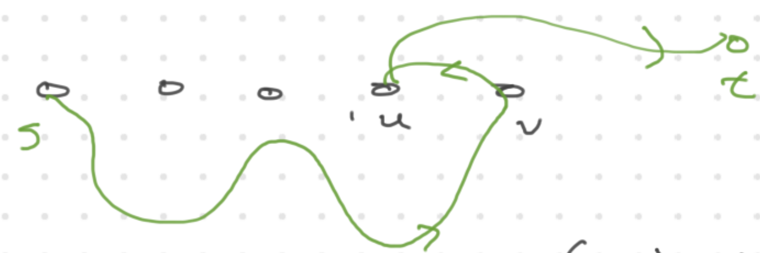
(because a shortest path linking x + y is also a shortest path from $x \to z$ for any intermediate vertex z)

$$dist_{G_i}(s,v) \le dist_{G_i}(s,u)+1$$
$$\le dist_{G_{i+1}}(s,u)+1$$
$$= dist_{G_{i+1}}(s,v)$$

contradicting our choice of v. - proving claim.

what does it mean that $(u,v) \notin E(G_i)$ but $(u,v) \in E(G_{i+1})$. So if we augmented on path Q from $s \to t$ to go from $G_i \to G_{i+1}$, Then $(v,u) \in E(Q)$ in order to reduce the flow on edge (u,v) so that (u,v) appears in G_{i+1}.

$$dist_{G_i}(s,v) = dist_{G_i}(s,u) - 1$$



$$\le dist_{G_{i+1}}(s,u) - 1$$
$$= dist_{G_{i+1}}(s,v) - 2$$

$$dist_{G_{i+1}}(s,v) < dist_{G_i}(s,v)$$
(because v fails lemma)

$$< dist_{G_{i+1}}(s,v).$$
(as shown above)

$dist_{G_{i+1}}(s,u) = dist_{G_{i+1}}(s,v) - 1$

so subbing in gives the equality. $\rightarrow\leftarrow$ ∎

Thm The total # of augmentations performed by Edmonds-Karp is $O(n \cdot m)$

Pf ~~agai~~ Let $G_0 = G$ + G_i, $i = 1, \dots, k$ be the residual graph after the i^{th} flow augmentation. Let f_i be the flow after the i^{th} augmentation w/ $f_0 = \overline{0}$. Note that each G_i is a subgraph of G_0 w/ $V(G_i) = V(G_0)$

Def an edge (u,v) of G to be critical at i if

$$(u,v) \in E(G_i) \text{ and } (u,v) \notin E(G_{i+1})$$

Def P_i to be the $s \to t$ path we augment on to go from $G_i \to G_{i+1}$

(u,v) is critical at i

$\iff$ $(u,v) \in E(P_i)$ AND

$$c(u,v) - f_i(u,v) = \min_{(x,y) \in E(P)} c(x,y) - f_i(x,y)$$

ie $\nexists$ edges of P_i are augmented by residual capacity of edge (u,v)

Note $\forall\, i$ $\exists$ at least one edge which is critical at i

Cl let $(u,v) \in E(G)$. Then $\exists$ at most $n/2$ distinct indices

$$\pi(1) < \pi(2) < \dots < \pi(\ell) \quad \text{ST}$$

(u,v) is critical at $\pi(i)$

pf Since $P_{\pi(i)}$ is a shortest s-t path in G_i $\Rightarrow$

$$dist_{G_i}(s,v) = dist_{G_i}(s,u) + 1$$

The edge $(u,v) \notin E(G)_{\pi(i)+1}$ so in order for (u,v) to be critical in $G_{\pi(i+1)}$ $\Rightarrow$

$\Rightarrow \exists\ i' \quad \pi(i) < i' < \pi(i+1)$ ST $(u,v) \notin G_{i'}$ but is an element of $G_{i'+1}$

$\Rightarrow (v,u) \in E(P_{i'})$

$$dist_{G_{i'}}(s,u) = dist_{G_{i'}}(s,v) + 1$$

so since $dist_{G_i}$ is monotonically increasing

$$\begin{aligned} dist_{G_{i'}}(s,u) &= dist_{G_{i'}}(s,v)+1 \\ &\geq dist_{G_{\pi(i)}}(s,v)+1 \\ &= dist_{G_{\pi(i)}}(s,u)+2 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} dist_{G_{\pi(i+1)}}(s,u) &\geq dist_{G_{i'}}(s,u) \\ &\geq dist_{G_{\pi(i)}}(s,u)+2 \end{aligned}$$

The distance to tail of edge (u,v)

ie distance to u must increase

by ≥ 2 ~~for~~ at each of

$\pi(1)$, $\pi(2)$, $\pi(3)$, ... etc.

+ since at the end, ~~max~~ $dist_{G_k}(s,u)$

is at most $n-1$ $\Rightarrow$ $\ell \leq n/2 - 1$

proving the claim ✓

+ now the Theorem follows easily:

every edge is critical at most $n/2$

times, there are m edges, +

for every i, some edge is critical

in G_i $\Rightarrow$ $k \leq n/2 \cdot m$ by pigeon hole

as desired. ∎

Conclusion

Complexity of

Edmonds-Karp $= O(\underbrace{n \cdot m}_{\text{\# flow augmentations}} \underbrace{(n+m)}))$

each flow augmentation

takes $O(n+m)$: $O(n+m)$

to find residual graph +

$O(n+m)$ to find shortest

path from $s \to t$ via BFS.

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improved to 2010 $O(n \cdot m)$